The Fokker-Planck Equation: Transporting Probabilities

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These notes are meant to provide a physical intuition for the Fokker-Planck equation by exploiting its similarity to the advection-diffusion equation.

Continuity Equation

Let's start with the continuity equation which essentially captures the (seemingly trivial) statement "stuff does not simply vanish" mathematically.

Consider an arbitrary quantity with volume density $\rho = \rho(\boldsymbol{x}, t)$, e.g., a dye dissolved in water. The total amount of the quantity in any given volume V is $\int_V \rho \, dV$. Excluding sources, sinks, and teleportation, the only possibility for the total amount to change is a flux of the quantity out of the surface S of the volume. The *flux* $\boldsymbol{j} = \boldsymbol{j}(\boldsymbol{x}, t)$ describes the change of the quantity per (infinitesimal) unit of time and (infinitesimal) area, thus we get the total change of the quantity by integrating the flux over the surface area:

$$\partial_t \int_V \rho \, dV = -\int_S \boldsymbol{j} \cdot d\boldsymbol{S}. \tag{1}$$

This is the continuity equation in integral form. Using the divergence theorem, or Gauss's theorem, $\int_S \boldsymbol{j} \cdot d\boldsymbol{S} = \int_V (\nabla \cdot \boldsymbol{j}) \, dV$ and the fact that the volume under consideration is arbitrary, we arrive at the *continuity equation* in differential form:

$$\partial_t \rho = -\nabla \cdot \boldsymbol{j}. \tag{2}$$

Note that the only assumption entering the derivation is the absence of sources, sinks, and teleportation. Thus, the continuity equation holds for any locally conserved quantity: mass, energy, electrical charge, momentum, ...

Advection-Diffusion Equation

First, let us consider the situation where the quantity is transported by an externally imposed flow field $\boldsymbol{u} = \boldsymbol{u}(\boldsymbol{x}, t)$, e.g., a dye in a river. For a more complicated example, see this video of a dye in a vortex. The *advective flux* is determined by the flow field:

$$\boldsymbol{j}_{\mathrm{adv}} = \boldsymbol{u}\rho. \tag{3}$$

In combination with the continuity equation (2), we get the *advection equation*

$$\partial_t \rho = -\nabla \cdot (\boldsymbol{u}\rho). \tag{4}$$

If we knew the flow field of the vortex in the above video, plugging it into Eq. (4) in combination with an appropriate initial distribution would yield the awesome dynamics of the dye.

Next, let us take diffusion into account. According to *Fick's law*, the diffusive flux is caused by a gradient in the density $\nabla\rho$,

$$\boldsymbol{j}_{\mathrm{Fick}} = -D\nabla\rho. \tag{5}$$

Here, the *diffusivity* $D = D(\boldsymbol{x}, t)$ is in the most general form a symmetric, positive definite matrix. If the diffusivity is a scalar the medium is isotropic; if the diffusivity does not depend on the position the medium is homogeneous. Plugging the diffusive flux into the continuity equation, we get the diffusion equation

$$\partial_t \rho = \nabla \cdot (D \nabla \rho). \tag{6}$$

See Fig. 5.12 on p. 140 in the soil physics lecture notes by Kurt Roth for an example solution of the stationary diffusion equation $\nabla \cdot (D \nabla \rho) = 0$ driven by the boundary conditions in a heterogeneous medium (the diffusivity is shown in Fig. 5.11 on p. 138).

Finally, combining advective flux and diffusive flux, $\boldsymbol{j} = \boldsymbol{j}_{\mathrm{adv}} + \boldsymbol{j}_{\mathrm{Fick}}$, and the continuity equation we arrive at the *advection-diffusion equation*

$$\partial_t \rho = -\nabla \cdot (\boldsymbol{u} \rho) + \nabla \cdot (D \nabla \rho). \tag{7}$$

It describes the transport of a quantity due to advection and diffusion. For an interesting example of the interplay between advection and diffusion, let us briefly consider *Taylor dispersion* based on Fig. 4.2 on p. 106 in the soil physics lecture notes. Another example is Fig. 7.11ff on p. 246ff.

Fokker-Planck Equation

The advection-diffusion equation (7) resembles the *Fokker-Planck equation* [1, Eq. 1.16]

$$\partial_t p = -\partial_i D_i^{(1)} p + \partial_i \partial_j D_{ij}^{(2)} p. \tag{8}$$

Here, $D_i^{(1)}$ is the *drift vector* and $D_{ij}^{(2)}$ the *diffusion matrix*. Note that we use the sum convention.

At first glance, the correspondence is $p = \rho$, $D_i^{(1)} = u_i$, and $D_{ij}^{(2)} = D_{ij}$. This makes sense: the quantity under consideration is the probability which is locally conserved, the drift acts as an externally imposed flow field, and the diffusion matrix equals the diffusivity. If the latter is independent of the position, this correspondence is indeed exact. Thus, in this case, the Fokker-Planck equation is the advection-diffusion equation for the probability density.

In the general case, the situation is a bit more complicated. To see this, let us rewrite Eq. (7) into a Fokker-Planck equation,

$$\partial_t \rho = -\partial_i u_i \rho + \partial_i D_{ij} \partial_j \rho = -\partial_i [u_i + (\partial_j D_{ij})] \rho + \partial_i \partial_j D_{ij} \rho, \tag{9}$$

where we used the product rule in the last step. We see that a *spurious drift* term appears, i.e., the correct correspondence of the drift term is $D_i^{(1)} = u_i + \partial_j D_{ij}$.

This is akin to the spurious drift depending on the interpretation of an underlying SDE. For the SDE $\dot{x}_i = h_i + g_{ij} \xi_j$ where $\langle \xi_i(t) \rangle = 0$ and $\langle \xi_i(t) \xi_j(t') \rangle = 2\delta_{ij} \delta(t - t')$ we have [2]

$$D_i^{(1)} = h_i + 2\lambda g_{kj} \partial_k g_{ij}, \qquad D_{ij}^{(2)} = g_{ik} g_{jk}, \tag{10}$$

where $\lambda \in [0, 1]$ determines the interpretation: $\lambda = 0$ for Itô, $\lambda = 1/2$ for Stratonovich, and $\lambda = 1$ for Hänggi-Klimontovich (or kinetic). For the special case $g_{ij} = g_i \delta_{ij}$ the diffusion matrix is diagonal, $D_{ij}^{(2)} = g_i^2 \delta_{ij}$, and we can rewrite the drift as $D_i^{(1)} = h_i + \lambda \partial_j D_{ij}^{(2)}$. Thus, if we choose the Hänggi-Klimontovich interpretation $\lambda = 1$, we get the direct correspondence $u_i = h_i$ and $D_{ij} = g_i^2 \delta_{ij}$. Put differently, the Hänggi-Klimontovich interpretation corresponds to the advection-diffusion equation. However, for arbitrary g_{ij} this correspondence does not hold (as far as I can see) because $2 g_{kj} \partial_k g_{ij} \neq \partial_j g_{ik} g_{jk}$.

REFERENCES

[1] H. Risken, *The Fokker-Planck Equation* (Springer Verlag Berlin Heidelberg, 1996).

[2] R. L. Stratonovich, Some markov methods in the theory of stochastic processes in nonlinear dynamical systems, in *Noise in Nonlinear Dynamical Systems*, Vol. 1, edited by F. Moss and P. V. E. McClintock (Cambridge University Press, 1989) pp. 16–71.